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Weak and Strong Duality on Optimization Problems with *n*-set Functions

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Abstract **-** In the present paper we develop some duality results for mathematical programming problems with differentiable convex n - set functions. Our main objective is to prove, the (i) weak duality theorem, (ii) strong duality theorem, and (iii) Mangasarian's strict converse duality theorem for a minimization problem with differentiable convex n - set functions and its dual problem.

Keywords: *Duality*, *Differentiable*, *Convex*, *n – set functions*, *Objective*, *Minimization*

I. INTRODUCTION

Our main focus in this paper is to develop the duality theory for nonlinear programming involving n - set functions. Some early work has been done on duality theory by Corley in [1] who developed the general theory of n - set functions. Corely obtained saddle point optimality conditions and also Lagrangian duality results for the problem (MP) given below. Ref. [2], Zalmai presented several duality results under generalized ρ -convexity assumptions for the same (MP) problem.

Ref. [3], Bector, Bhatia, and Pandey considered a class of multiobjective programming problems with differentiable n - set functions and established duality results and later in [4], Bector, Bhatia, and Pandey obtained duality results for a nonlinear miltiobjective fractional programming problem.

In this paper our aim is to develop weak duality, strong duality, and Mangasarian strict converse duality results for the minimization problem (MP) with differentiable convex n-set functions.

Let A^n be a family and let F and G be, respectively, convex n - set functions and convex m - dimensional n - set functions, both defined on A^n.

The (primal) minimization problem (MP) is defined as follows:

$$\text{minimize } F\ (R_1,...R_n)$$

subject to (MP)

$$(R_1,...,R_n) \in S^{n},$$

where

$$S^n = \{(R_1,..., R_n) \in A^n : G_j(R_1,...R_n) \le 0,\ j = 1,..., m\}\cdot$$

The (dual) maximization problem (DP) of the (MP) is defined as follows:

$$\text{maximize } F\ (S_1,...S_n) + \sum_{j=1}^{m} u_j G_j (S_1,...S_n),$$

subject to

$$\left\langle f^i{}_s + \sum_{j=1}^{m} u_j g^i{}_{js}, \chi_{Ri} - \chi_{Si} \right\rangle \;\geq\; 0, \text{ for all } R_i \in A,\ i = 1,...n, \qquad \text{(DP)}$$

$$(S_1,...S_n) \;\in\; A^n,$$

$$u_j \;\geq\; 0,\ j = 1,...,m,$$

$f^i{}_s$ and $g^i{}_{jS}$ respectively the i th partial derivatives of F and G_j at $(S_1,...S_n)$. Denote by Γ the set of all feasible solutions $(S_1,...S_n; u_1,...u_m)$ which satisfy the constraints of (DP).

II. RESULTS AND DISCUSSION

Theorem 1 below, referred as the weak duality theorem, shows that the objective function value of any feasible solution to the dual problem yields a lower bound on the objective function value of any feasible solution to the primal problem.

Weak duality theorem for n-set functions:

Theorem 1. *Let A^n be a subfamily and let F and G be differentiable on A^n. If $(S_1,...S_n) \in S^n$, $(R_1,..., R_n; u_1,..., u_m) \in \Gamma$, and F and G are convex at $(R_1,...R_n)$, then*

$$F(S_1,..., S_n) \;\geq\; F(R_1,..., R_n) + \sum_{j=1}^{m} u_j G_j (R_1,..., R_n).$$

Proof. Since F is convex and differentiable, by Theorem 4.5 in [1], we have

$$F(S_1,...,S_n) \;\geq\; F(R_1,...R_n) + \sum_{i=1}^{n} \left\langle f^i{}_R,\ \chi_{Si} - \chi_{R_i} \right\rangle$$

$$\geq\; F(R_1,...R_n) - \sum_{i=1}^{n} \left\langle \sum_{j=1}^{m} u_j g^i{}_{jR}, \chi_{Si} - \chi_{R_i} \right\rangle$$

(since $(R_1,..., R_n; u_1,...u_m) \in \Gamma$)

$$\geq\; F(R_1,..., R_n) + \sum_{j=1}^{m} u_j (G_j (R_1,..., R_n) - G_j (S_1,..., S_n))$$

(by Theorem 4.5 in [1])

$$\geq\; F(R_1,..., R_n) + \sum_{j=1}^{m} u_j G_j (R_1,..., R_n).$$

(since $u_j G_j (S_1,..., S_n) \leq 0, j = 1,..., m$).

This concludes the proof of the theorem.

Theorem 2 below, referred as the strong duality theorem, shows that under suitable convexity assumptions, the optimal objective function values of the primal and dual problems are equal. This theorem is considered to be one of the more important duality theorems of nonlinear programming.

Strong duality theorem for n-set functions:

Theorem 2. *Let A^n be a family, F and G differentiable convex $n-$set functions on A^n, and let $(\hat{S}_1,...,\hat{S}_n)$ solve (MP). If G satisfies the following Kuhn–Tucker conditions at $(\hat{S}_1,...,\hat{S}_n)$, that is, there exist nonnegative $\hat{u}_1,...,\hat{u}_m$ such that*

$$\left\langle f^i{}_s + \sum_{j=1}^{m} \hat{u}_j g^i{}_{j\hat{s}},\ \chi_{R_i} - \chi_{\hat{S}_i} \right\rangle \geq 0 \text{ for all } R_i \in A,\ i = 1,...,n$$

$$\sum_{j=1}^{m} \hat{u}_j G_j(\hat{S}_1,...\hat{S}_n) = 0$$

$$G_j(\hat{S}_1,...,\hat{S}_n) \leq 0,\ j = 1,...,m$$

where $f^i{}_s$ and $g^i{}_{j\hat{s}}$ are respectively the i th partial derivatives of F and G_j at the point $(\hat{S}_1,...,\hat{S}_n)$, then $(\hat{S}_1,...,\hat{S}_n;\hat{u}_1,...\hat{u}_m)$ solves (DP) and

$$F(\hat{S}_1,...,\hat{S}_n) = F(\hat{S}_1,...,\hat{S}_n) + \sum_{j=1}^{m} \hat{u}_j\, G_j(\hat{S}_1,...,\hat{S}_n).$$

Proof. $F(\hat{S}_1,...,\hat{S}_n) + \sum_{j=1}^{m} \hat{u}_j\, G_j(\hat{S}_1,...,\hat{S}_n) = F(\hat{S}_1,...,\hat{S}_n)$

(since $\sum_{j=1}^{m} \hat{u}_j G_j(\hat{S}_1,...,\hat{S}_n) = 0$)

$$\geq F(R_1,...,R_n) + \sum_{j=1}^{m} G_j(R_1,...,R_n)$$

(by Theorem 1)

for all $(R_1,...,R_n;u_1,...,u_m) \in \Gamma$.
Hence the proof is complete.

Before we state the next theorem, we define the strict convexity of an n-set function.

Definition 3. *An $n-$set function $F : S^n \to R$ is said to be strictly convex on a convex subfamily S^n of A^n if for each $(R_1,...,R_n),(S_1,...,S_n) \in S^n$, $(R_1,...,R_2) \neq (S_1,...,S_n)$, and $\lambda \in [0,1]$,*

$$\limsup_{l\to\infty} F(V^l{}_1(\lambda),...,V^l{}_n(\lambda)) \langle \lambda F(R_1,...,R_n) + (1-\lambda)F(S_1,...,S_n)$$

for any Morris sequence $\{V^l{}_i(\lambda)\} \subset S$ associated with $\langle \lambda, R_i, S_i \rangle$ for each $i = 1,...,n$.

Before we represent the next duality theorem, we state the following lemma, which we need to prove the theorem.

Lemma 4. *Let $F : S^n \to R$ be differentiable on a convex subfamily S^n of A^n. If F*

is strictly convex, then for all $(R_1,\ldots,R_n),(S_1,\ldots,S_n)\in S^n$

$$F(R_1,\ldots,R_n)-F(S_1,\ldots,S_n) \;\rangle\; \sum_{i=1}^{n}\left\langle f^i{}_*,\chi_{R_i}-\chi_{S_i}\right\rangle,$$

where $f^i{}_*$ *is the* i *th partial derivative of* F *at* $(S_1,\ldots S_n)$.

This lemma can be proved by using Definition 4.3 and the proof of Theorem 4.5 in [1].

Another important duality theorem is the converse of the strong duality theorem. In order to obtain such a theorem we have to modify the hypothesis of the strong duality theorem. Theorem 5 below is such a theorem referred as a strict converse duality theorem. The theorem was originally introduced and proved by [5] for real functions defined on R^n. The following is an extension of that theorem for n-set functions.

Mangasarian's strict converse duality theorem:

Theorem 5. *Let* A^n *be a family,* F *and* G *be differentiable and convex on* A^n. *Let* $(\overline{S}_1,\ldots,\overline{S}_n)$ *be a solution of* (MP) *and assume that* G *satisfies the Kuhn–Tucker conditions at* $(\overline{S}_1,\ldots\overline{S}_n)$. *If* $(\hat{S}_1,\ldots,\hat{S}_n;\hat{u}_1,\ldots,\hat{u}_m)$ *is a solution of* (DP) *and if*

$$F(R_1,\ldots R_n)+\sum_{j=1}^{m}\hat{u}_j\,G_j(R_1,\ldots,R_n)$$

is strictly convex at $(\hat{S}_1,\ldots,\hat{S}_n)$, *then* $(\hat{S}_1,\ldots,\hat{S}_n)=(\overline{S}_1,\ldots,\overline{S}_n)$, *that is,* $(\hat{S}_1,\ldots,\hat{S}_n)$ *also solves* (MP), *and*

$$F(\overline{S}_1,\ldots,\overline{S}_n) \;=\; F(\hat{S}_1,\ldots\hat{S}_n)+\sum_{j=1}^{m}\hat{u}_j\,G_j(\hat{S}_1,\ldots,\hat{S}_n).$$

Definition 6. $F(R_1,\ldots,R_n)+\sum_{j=1}^{m}\hat{u}_j\,G_j(R_1,\ldots,R_n)$ is strictly convex at $(\hat{S}_1,\ldots,\hat{S}_n)$ if either F is strictly convex at $(\hat{S}_1,\ldots,\hat{S}_n)$ or if for some j, $\hat{u}_j \rangle 0$ and G_j is strictly convex at $(\hat{S}_1,\ldots,\hat{S}_n)$.

Proof. For simplicity we again let

$$\psi(R_1,\ldots,R_n;u_1,\ldots,u_m)=F(R_1,\ldots,R_n)+\sum_{j=1}^{m}u_jG_j(R_1,\ldots,R_n).$$

We shall assume that $(\hat{S}_1,\ldots,\hat{S}_n)\neq(\overline{S}_1,\ldots,\overline{S}_n)$ and exhibit a contradiction.

Since $(\overline{S}_1,\ldots,\overline{S}_n)$ is a solution of (MP), and G satisfies Kuhn-tucker conditions at $(\overline{S}_1,\ldots,\overline{S}_n)$, it follows from the previous theorem that there exists a $(\overline{u}_1,\ldots,\overline{u_m})\in R^m$ such that $(\overline{S}_1,\ldots,\hat{S}_n\;,\overline{u}_1,\ldots,\overline{u}_m)$ solves (DP).

Hence, $\psi(\overline{S}_1,\ldots,\overline{S}_n,\overline{u}_1,\ldots,\overline{u}_m)=\psi(\hat{S}_1,\ldots,\hat{S}_n;\hat{u}_1,\ldots,\hat{u}_m)=\max\psi(R_1,\ldots,R_n;u_1,\ldots,u_m)$,

over $(R_1,\ldots,R_n;u_1,\ldots,u_m)\in\Gamma$ and $(\overline{S}_1,\ldots,\overline{S}_n,\hat{u}_1,\ldots,\hat{u}_m)\in\Gamma$.

Let us define $\psi^i{}_{\hat{S},\hat{u}}=f^i{}_{\hat{S}}+\sum_{j=1}^{m}u_jg^i{}_{j\hat{S}}$, the i^{th} partial derivative at

$(\hat{S}_1,\ldots,\hat{S}_n;\hat{u}_1,\ldots,\hat{u}_m)\in\Gamma$.

Because $(\hat{S}_1,\ldots,\hat{S}_n;\hat{u}_1,\ldots,\hat{u}_m)\in\Gamma$, we have that,

$$\sum_{i=1}^{n}\left\langle \psi^{i}{}_{\hat{S},\hat{u}},\ \chi_{\bar{S}_i}-\chi_{\hat{S}_i}\right\rangle \geq 0 \quad \text{for } \bar{S}_i\in A,\ i=1,..,n.$$

Hence, by the strict convexity of $\psi(R_1,\ldots,R_n;\hat{u}_1,\ldots,\hat{u}_m)$ at $(\hat{S}_1,\ldots,\hat{S}_n)$ and by the Lemma 4 it follows that

$$\psi(\bar{S}_1,\ldots,\bar{S}_n,\hat{u}_1,\ldots,\hat{u}_m)-\psi(\hat{S}_1,\ldots,\hat{S}_n;\hat{u}_1,\ldots,\hat{u}_m) > \sum_{i=1}^{n}\left\langle \psi^{i}{}_{\hat{S},\hat{u}},\ \chi_{\bar{S}_i}-\chi_{\hat{S}_i}\right\rangle \geq 0.$$

As a consequence

$$\psi(\bar{S}_1,\ldots,\bar{S}_n,\hat{u}_1,\ldots,\hat{u}_m) > \psi(\hat{S}_1,\ldots,\hat{S}_n;\hat{u}_1,\ldots,\hat{u}_m)=\psi(\bar{S}_1,\ldots,\bar{S}_n;\bar{u}_1,\ldots,\bar{u}_m).$$

That is, $\sum_{j=1}^{m}\hat{u}_j\, G_j(\bar{S}_1,\ldots,\bar{S}_n) > \sum_{j=1}^{m}\bar{u}_j G_j(\bar{S}1,\ldots,\bar{S}_n)$.

But $\sum_{j=1}^{m}\bar{u}_j G_j(\bar{S}_1,\ldots,\bar{S}_n)=0$ (Kuhn-Tucker condition), hence

$$\sum_{j=1}^{m}\hat{u}_j\, G_j(\bar{S}_1,\ldots,\bar{S}_n) > 0.$$

This contradicts the facts that $\hat{u}_j\geq 0$ and $G_j(\bar{S}_1,\ldots,\bar{S}_n)\leq 0$ for each $j=1,\ldots,m.$

Hence, $(\hat{S}_1,\ldots,\hat{S}_n)=(\bar{S}_1,\ldots,\bar{S}_n)$.

It is also the case that

$$\begin{aligned} F(\bar{S}_1,\ldots,\bar{S}_n) &= F(\bar{S}_1,\ldots,\bar{S}_n)+\sum_{j=1}^{m}\bar{u}_j G_j(\bar{S}_1,\ldots,\bar{S}_n) \\ &= \psi(\bar{S}_1,\ldots,\bar{S}_n,\bar{u}_1,\ldots,\bar{u}_m) \\ &= \psi(\hat{S}_1,\ldots,\hat{S}_n;\hat{u}_1,\ldots,\hat{u}_m). \end{aligned}$$

Therefore, $F(\bar{S}_1,\ldots,\bar{S}_n) = F(\hat{S}_1,\ldots\hat{S}_n)+\sum_{j=1}^{m}\hat{u}_j\, G_j(\hat{S}_1,\ldots,\hat{S}_n).$

The proof of the theorem is now complete.

III. CONCLUSION

In this paper, we concentrated on duality theorems for n-set functions. We considered a minimization problem and its dual problem and obtained weak duality, strong duality, and Mangasarian strict converse duality results. Here, we assumed that n-set functions are convex and differentiable.

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